

Impulse Control of a Diffusion with a Change Point

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Abstract

This paper solves a Bayes sequential impulse control problem for a diffusion, whose drift has an unobservable parameter with a change point. The partially-observed problem is reformulated into one with full observations, via a change of probability measure which removes the drift. The optimal impulse controls can be expressed in terms of the solutions and the current values of a Markov process adapted to the observation filtration. We shall illustrate the application of our results using the Longstaff-Schwartz algorithm for multiple optimal stopping times in a geometric Brownian motion stock price model with drift uncertainty.

Keywords Bayes sequential optimization; impulse control; change point; change of measure; Longstaff-Schwartz algorithm

1 Introduction

Suppose the price evolution of a stock follows a geometric Brownian motion, whose drift will change at an unknown future time to an unknown level. An investor, who purchases a certain small number of shares at an initial time, can only observe the evolution of prices. Based on these observed prices and some reasonable *a priori* knowledge about the change in the drift, what is the best time to sell the shares, in order to maximize the investor's expected profit? Under the same circumstances, what about the optimal discretely-balanced buying/selling trading strategies with a larger number of shares?

At a more abstract level, this is a question about optimal stopping and impulse control of a diffusion, whose drift term has an unobservable parameter with a change

point. There have been three common approaches to such problems. The conservative approach is the mini-max philosophy that optimizes the worst-case scenario, formulated as a zero-sum game between a controller and a stopper by [20] Karatzas and Zamfirescu (2008). The approach employed by many practitioners is to divide model calibration and decision making into two separate steps. Another approach is to convert the decision-making problem with partial observations, into one with full observations, by augmenting the state process with the posterior probability distribution of the parameter. One illustration of this method is the work [10] Dai, Zhang and Zhu (2010). In a geometric Brownian motion price model with drift uncertainty, the authors assume no impact of the trading activities, and find two optimal sequences of times to place, respectively, buy and sell orders.

The topics mentioned in the previous paragraph are all very well developed fields of research with an extensive literature from the past decades: among them, [32] Shiryaev (1969) and [17] Karatzas (2003) for sequential detection; [33] Shiryaev (1978) or Appendix D in [19] Karatzas and Shreve (1998) for optimal stopping problems; [3], [4], [5] and [6] by Bensoussan and Lions and [28] Øksendal and Sulem (2007) for impulse controls; as well as [26] Liptser and Shiryaev (2001) and [7] Bensoussan (1992) for solving partially observed control problems using filtering techniques.

This paper is an attempt at solving impulse control problems in the Bayes sequential framework in one step, without tracking the posterior probability processes. The conversion from partial to full observations is facilitated by a change of probability measure, which hides the drift part of the diffusion. The measure change method was originally developed for solving change-point detection problems. For our problem, the state process augmented by the likelihood ratios is Markovian under the reference probability measure, and one can derive the dynamic programming principle satisfied by the value functions. The current values of the augmented state process provide all the information necessary for decision making.

There are at least three widely used methods for describing the value functions of stochastic control problems - PDE, dynamic programming, and backward SDE. They are different formulations of the notion of the “stochastic maximum principle” (c.f. [23] Kushner (1972) and [11] Davis (1973)). The common mechanism of all three methods is that the sum of the value function and the cumulative reward, when evaluated along the state process corresponding to any admissible control strategy, yields a supermartingale – which becomes a martingale if and only if the control strategy is optimal. After reduction to the Markovian case, we represent the optimal impulse control via the dynamic programming principle. Unlike in the pioneering papers [3], [4], [5], [6] and [7] of Bensoussan and Lions, the variational inequalities associated with the value functions will not be presented here, because they are numerically inefficient to implement due to the dimensionality of the augmented state process. From a numerical point of view, we adapt the Longstaff-Schwartz algorithm for multiple optimal stopping times. Based on the formulation of the dynamic programming principle in terms of stopping times, this method is computationally efficient although it uses Monte Carlo simulation. We also show the good

accuracy of the obtained results for a simple problem involving geometric Brownian motion and two optimal stopping times. The reduction to full observation via the change of measure and the proposed algorithm that involves Monte Carlo are well suited to a high dimensional state process. They are a contribution to the methodology of solving partial observation control problems.

In Section 2, we specify the model and the quantity that we want to optimize. Section 3 deals with the main theoretical results related to writing and solving the impulse control problem under a different probability measure. Section 4 illustrates how to use the theoretical results through the implementation of Longstaff-Schwartz algorithm for multiple optimal stopping times. Finally, we show in Section 5 that the method presented in this paper provides a better framework for multidimensional problems than the usual one that involves the posterior probabilities.

2 Problem formulation

This section sets up the diffusion model with a change point in its drift with Section 2.1 and lays out the impulse control problem of the diffusion with Section 2.2.

2.1 The diffusion with a change point

We consider a probability space $(\Omega, \mathbb{F}, \mathbb{P}^0)$, which supports two independent, \mathbb{F} -measurable random variables ρ and U , as well as a one-dimensional standard Brownian motion $W^0(\cdot)$ with respect to its natural filtration \mathbf{F}^{W^0} . The vector of random variables (ρ, U) is independent of the Brownian motion $W^0(\cdot)$. Let

$$\mathbf{G} = \{\mathbf{G}(t)\}_{0 \leq t \leq T} = \sigma \{\rho, U, W^0(s); 0 \leq s \leq t\}_{0 \leq t \leq T} \quad (2.1)$$

denote the filtration generated by ρ , U and $W^0(\cdot)$.

The unobservable process

$$\begin{aligned} \theta : [0, T] \times \Omega &\rightarrow \Theta, \\ (t, \omega) &\mapsto \theta(t, \omega) =: \theta(t) \end{aligned} \quad (2.2)$$

takes values in the parameter space $\Theta = \{\mu_0, \mu_1, \dots, \mu_m\} \subset \mathbb{R}$. The process $\theta(\cdot)$ starts with initial value $\theta(0) = \mu_0$, and keeps this value until an unobservable time ρ of *regime change*. At the time ρ , the parameter $\theta(\cdot)$ changes to a new level U , a random variable taking values in the set $\{\mu_1, \dots, \mu_m\}$, and remains at that level until the fixed terminal time $T \in (0, \infty)$. If regime change does not occur by time T , then $\theta(\cdot)$ takes the value μ_0 throughout the interval $[0, T]$, namely,

$$\theta(t) = \begin{cases} \mu_0, & 0 \leq t < \rho \wedge T; \\ U, & \rho \wedge T \leq t \leq T. \end{cases} \quad (2.3)$$

The change point ρ and the level U have prior distributions

$$\mathbb{P}(\rho > t) = e^{-\lambda t}, \quad t \geq 0, \quad (2.4)$$

and

$$\mathbb{P}(U = \mu_j) = p_j, \quad j = 1, 2, \dots, m. \quad (2.5)$$

For any possible values $u \in \Theta$, the given measurable functions $b(\cdot, \cdot; u) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma(\cdot, \cdot) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following Lipschitz and boundedness condition.

Assumption 2.1 *There exists a constant $C > 0$, such that*

(i) for all $(t, x^1), (t, x^2) \in [0, T] \times \mathbb{R}$, and for all $u \in \Theta$, we have

$$|b(t, x^1; u) - b(t, x^2; u)| + |\sigma(t, x^1) - \sigma(t, x^2)| + \left| \frac{b(t, x^1; u)}{\sigma(t, x^1)} - \frac{b(t, x^2; u)}{\sigma(t, x^2)} \right| \leq C|x^1 - x^2|; \quad (2.6)$$

whereas

(ii) for all $(t, x) \in [0, T] \times \mathbb{R}$ and all $u \in \Theta$, we have

$$\sigma(t, x) > 0 \quad \text{and} \quad \left| \frac{b(t, x; u)}{\sigma(t, x)} \right| \leq C. \quad (2.7)$$

The Assumption 2.1 (i) implies a linear growth condition on the functions $b(t, \cdot; u)$ and $\sigma(t, \cdot)$: there exists another constant $C' > 0$, such that

$$|b(t, x; u)| + |\sigma(t, x)| \leq C'|1 + x| \quad (2.8)$$

holds for all $u \in \Theta$ and all $(t, x) \in [0, T] \times \mathbb{R}$.

Let N be a positive integer, $0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_N \leq T$ be stopping times with respect to the filtration \mathbf{F}^{W^0} , and ζ_i be an \mathbb{R} -valued $\mathbf{F}^{W^0}(\tau_i -)$ -measurable random variable for $i = 1, 2, \dots, N$. The N -tuple $(\tau, \zeta) = \{(\tau_i, \zeta_i)\}_{i=1}^N$ is called an *impulse control*. The set of *admissible controls*, denoted as \mathbf{I} , is the collection of all such impulse controls (τ, ζ) . The jump size $\gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given bounded, measurable function. Given an arbitrary impulse control $(\tau, \zeta) \in \mathbf{I}$, the controlled state process $X(\cdot)$ is the unique strong solution to the equation

$$X(t) = x_0 + \int_0^t \sigma(s, X(s)) dW^0(s) + \sum_{\tau_i \leq t} \gamma(X(\tau_i -), \zeta_i), \quad 0 \leq t \leq T. \quad (2.9)$$

By Assumption 2.1 (ii) and equation (2.9), the Brownian filtration \mathbf{F}^{W^0} coincides with $\mathbf{F} = \{\mathbf{F}(t)\}_{0 \leq t \leq T}$, which denotes the filtration generated by the process $X(\cdot)$. The collection of all \mathbf{F} -stopping times with values in $[0, T]$ is denoted as \mathbf{S} , and the collection of all \mathbf{F} -stopping times with values in $[t, T]$ is denoted as \mathbf{S}_t .

• In order to define another probability measure \mathbb{P} on the space Ω and on the sigma algebra $\mathbf{G}(T)$, the one with respect to which we shall formulate our impulse control problem, we introduce the \mathbf{G} -adapted process

$$Z(t) = \exp \left\{ \int_0^t \frac{b(s, X(s); \theta(s))}{\sigma(s, X(s))} dW^0(s) - \frac{1}{2} \int_0^t \frac{b^2(s, X(s); \theta(s))}{\sigma^2(s, X(s))} ds \right\}, \quad 0 \leq t \leq T, \quad (2.10)$$

which will play the role of Radon-Nikodym derivative of the new measure \mathbb{P} with respect to the “reference probability measure” \mathbb{P}^0 . Whereas, for every number $u \in \Theta$, the \mathbf{F} -adapted likelihood ratio process is defined as

$$L(t; u) = \exp \left\{ \int_0^t \frac{b(s, X(s); u)}{\sigma(s, X(s))} dW^0(s) - \frac{1}{2} \int_0^t \frac{b^2(s, X(s); u)}{\sigma^2(s, X(s))} ds \right\}, \quad 0 \leq t \leq T. \quad (2.11)$$

From the expression (2.3) for $\theta(\cdot)$, the Radon-Nikodym derivative $Z(\cdot)$ can be written, in terms of the likelihood ratio process $L(\cdot; u)$ and of the random vector (ρ, U) , as

$$Z(t) = L(\rho; \mu_0) \left(\sum_{j=1}^m \mathbf{1}_{\{U=\mu_j\}} \frac{L(t; \mu_j)}{L(\rho; \mu_j)} \right) \mathbf{1}_{\{\rho < t\}} + L(t; \mu_0) \mathbf{1}_{\{\rho \geq t\}}, \quad 0 \leq t \leq T. \quad (2.12)$$

The Radon-Nikodym process $Z(\cdot)$ in (2.10) is a $(\mathbb{P}^0, \mathbf{G})$ -martingale, because of Assumption 2.1 (ii) on the boundedness of the ratio $b(\cdot, \cdot; u)/\sigma(\cdot, \cdot)$ and of the Novikov condition; the same is true for the likelihood ratio process $L(\cdot; u)$ in (2.11), for any $u \in \Theta$. There exists then a probability measure \mathbb{P} equivalent to \mathbb{P}^0 , satisfying

$$\left. \frac{d\mathbb{P}}{d\mathbb{P}^0} \right|_{\mathbf{G}(t)} = Z(t), \quad 0 \leq t \leq T. \quad (2.13)$$

Under this new probability measure \mathbb{P} , the random variables ρ and U are still independent and retain the prior distributions of (2.4) and (2.5). By a generalization of the Girsanov theorem to local martingales in [36] Van Schuppen and Wong (1974), the process

$$\left\{ \int_0^t \sigma(s, X(s)) dW^0(s) - \int_0^t b(s, X(s); \theta(s)) ds \right\}_{0 \leq t \leq T} \quad (2.14)$$

is a local (\mathbb{P}, \mathbf{G}) -martingale, having the instantaneous quadratic variation $\sigma^2(\cdot, X(\cdot))$; and the process $W(\cdot)$ defined as

$$W(t) := W^0(t) - \int_0^t \sigma^{-1}(s, X(s)) b(s, X(s); \theta(s)) ds, \quad 0 \leq t \leq T \quad (2.15)$$

is a standard \mathbb{P} -Brownian motion. The process $X(\cdot)$ defined by (2.9) is also the unique strong solution to the equation

$$X(t) = x_0 + \int_0^t b(s, X(s); \theta(s)) ds + \int_0^t \sigma(s, X(s)) dW(s) + \sum_{\tau_i \leq t} \gamma(X(\tau_i-), \zeta_i), \quad 0 \leq t \leq T. \quad (2.16)$$

2.2 Impulse control of the diffusion

The impulse control problem we study in this paper, consists in choosing an optimal impulse control $(\tau^*, \zeta^*) = \{(\tau_i^*, \zeta_i^*)\}_{i=1}^N$ to achieve the maximal expected reward

$$V := \sup_{(\tau, \zeta) \in \mathbf{I}} \mathbb{E} \left[\int_0^T h(X(t)) dt + \xi(X(T)) + \sum_{i=1}^N c(X(\tau_i-), \zeta_i) \right], \quad (2.17)$$

over all admissible impulse controls $(\tau, \zeta) = \{(\tau_i, \zeta_i)\}_{i=1}^N$ in \mathbf{I} . The reward functions ξ and $h : \mathbb{R} \rightarrow \mathbb{R}$ are measurable and satisfy conditions (i) and (ii) in Assumption 2.2 below. Furthermore, we impose growth condition on the deterministic measurable functions γ and $c : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ in the state variable, as follows.

Assumption 2.2 (i) *The function $\xi(\cdot)$ is twice continuously differentiable, with first and second order derivatives denoted as $\xi'(\cdot)$ and $\xi''(\cdot)$.*
(ii) *The functions $h(\cdot)$, $\xi(\cdot)$, $\xi'(\cdot)$ and $\xi''(\cdot)$ are locally Lipschitz and have polynomial growth.*
(iii) *The function $\gamma(x, z)$ is bounded for all $x \in \mathbb{R}$ and $z \in \mathbb{R}$, and the function $c(x, z)$ has polynomial growth rate in $x \in \mathbb{R}$ uniformly for all $z \in \mathbb{R}$. Both functions $\gamma(x, z)$ and $c(x, z)$ are continuous in x , for any arbitrarily fixed $z \in \mathbb{R}$.*

3 Solving the impulse control problem

This section provides a theoretical solution to the impulse control problem (2.17): Section 3.1 reduces the partially observable problem into one of full observation, by changing to the reference probability measure under which the state process is a martingale and augmenting the state process with the likelihood ratios and their integrals. Section 3.2 solves the fully observable impulse control problem under the reference probability measure, by representing the optimal control in terms of the value functions and the augmented state process.

3.1 The measure change method

By the “measure change method”, we mean considering the impulse control problem (2.17) under the reference probability measure \mathbb{P}^0 which removes the unobservable drift of the state process $X(\cdot)$. Using the Bayes rule and the properties of conditional expectations, the maximal expected reward V from (2.17) can be written as

$$\begin{aligned} V &= \sup_{(\tau, \zeta) \in \mathbf{I}} \mathbb{E}^0 \left[Z(T) \left(\int_0^T h(X(t)) dt + \xi(X(T)) + \sum_{i=1}^N c(X(\tau_i-), \zeta_i) \right) \right] \\ &= \sup_{(\tau, \zeta) \in \mathbf{I}} \mathbb{E}^0 \left[\mathbb{E}^0 [Z(T) | \mathbf{F}(T)] \left(\int_0^T h(X(t)) dt + \xi(X(T)) + \sum_{i=1}^N c(X(\tau_i-), \zeta_i) \right) \right]. \end{aligned} \quad (3.1)$$

From the Bayes point of view, the quantity $\mathbb{E}^0 [Z(t) | \mathbf{F}(t)]$ in (3.1) is the posterior expectation of the Radon-Nikodym derivative $Z(\cdot)$ under the reference probability measure \mathbb{P}^0 , given the observations of $X(\cdot)$ up to date. Because of the independence of (ρ, U) and $X(\cdot)$ under \mathbb{P}^0 , from the prior \mathbb{P}^0 -distributions (2.4) and (2.5), and by (2.12), this posterior expectation has the form

$$\mathbb{E}^0 [Z(t) | \mathbf{F}(t)] = \sum_{j=1}^m \left(p_j L(t; \mu_j) \int_0^t \frac{L(s; \mu_0)}{L(s; \mu_j)} \lambda e^{-\lambda s} ds \right) + e^{-\lambda t} L(t; \mu_0), \quad 0 \leq t \leq T. \quad (3.2)$$

For every $u \in \Theta$, the likelihood ratio process $L(\cdot; u)$ defined in (2.11) is a $(\mathbb{P}^0, \mathbf{F})$ -martingale satisfying the stochastic integral equation

$$L(t; u) = \int_0^t L(s; u) \frac{b(s, X(s); u)}{\sigma(s, X(s))} dW^0(s), \quad 0 \leq t \leq T, \quad (3.3)$$

with respect to the standard $(\mathbb{P}^0, \mathbf{F})$ -Brownian motion $W^0(\cdot)$. From equations (3.2) and (3.3) we obtain, for $0 \leq t \leq T$, that

$$\begin{aligned} d(\mathbb{E}^0[Z(t) | \mathbf{F}(t)]) &= \sum_{j=1}^m p_j \left(\int_0^t \frac{L(s; \mu_0)}{L(s; \mu_j)} \lambda e^{-\lambda s} ds \right) dL(t; \mu_j) + e^{-\lambda t} dL(t; \mu_0) \\ &= \left(\sum_{j=1}^m p_j \left(\int_0^t \frac{L(s; \mu_0)}{L(s; \mu_j)} \lambda e^{-\lambda s} ds \right) L(t; \mu_j) \frac{b(t, X(t); \mu_j)}{\sigma(t, X(t))} \right. \\ &\quad \left. + e^{-\lambda t} L(t; \mu_0) \frac{b(t, X(t); \mu_0)}{\sigma(t, X(t))} \right) dW^0(t), \end{aligned} \quad (3.4)$$

so the posterior expectation $\{\mathbb{E}^0[Z(t) | \mathbf{F}(t)]\}_{0 \leq t \leq T}$ is a local $(\mathbb{P}^0, \mathbf{F})$ -martingale; in fact a $(\mathbb{P}^0, \mathbf{F})$ -martingale, as is easily checked from the definition (2.13).

Applying Itô's formula, we shall see that the random variable inside the \mathbb{P}^0 -expectation on the second line of (3.1) is a $(\mathbb{P}^0, \mathbf{F})$ -semimartingale evaluated at the time T . Lemma 3.2 will show that its local martingale part is in fact a square-integrable $(\mathbb{P}^0, \mathbf{F})$ -martingale of class **D** (Definition 4.8, page 24 of [18] Karatzas and Shreve (1988)), because of the uniform integrability result from Lemma 3.1.

Lemma 3.1 *For every $u \in \Theta$, consider the process $R(\cdot; u)$ defined as*

$$R(t; u) := \int_0^t \frac{L(s; \mu_0)}{L(s; u)} \lambda e^{-\lambda s} ds, \quad 0 \leq t \leq T. \quad (3.5)$$

For any nonnegative integers q_1, q_2 and q_3 , we have

$$\mathbb{E}^0 \left[\sup_{0 \leq t \leq T} |X(t)|^{q_1} \right] < \infty, \quad \mathbb{E}^0 \left[\sup_{0 \leq t \leq T} L^{q_2}(t; u) \right] < \infty \quad \text{and} \quad \mathbb{E}^0 [R^{q_3}(T; u)] < \infty; \quad (3.6)$$

furthermore, the family

$$\left\{ \sup_{0 \leq t \leq T} |X(t)|^{q_1} L^{q_2}(\tau; u) R^{q_3}(\tau; u) \right\}_{\tau \in \mathcal{S}} \quad (3.7)$$

is uniformly integrable with respect to the probability measure \mathbb{P}^0 .

Proof: By Assumption 2.2 (iii) and equation (2.9), there exists $C_1(x_0, \gamma, N, q_1) \in (0, \infty)$, such that for any $0 \leq t \leq T$, we have

$$\begin{aligned} \sup_{0 \leq s \leq t} |X(s)|^{2q_1} &\leq \sup_{0 \leq s \leq t} \left(x_0 + \left| \int_0^s \sigma(w, X(w)) dW^0(w) \right| + \sum_{i=1}^N |\gamma(X(\tau_i-), \zeta_i)| \right)^{2q_1} \\ &\leq C_1(x_0, \gamma, N, q_1) \left(1 + \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(w, X(w)) dW^0(w) \right|^{2q_1} \right). \end{aligned} \quad (3.8)$$

Since $\int_0^\cdot \sigma(t, X(t))dW^0(t)$ is a local \mathbb{P}^0 -martingale, from the Burkholder-Davis-Gundy inequality (e.g. page 166 of [18] Karatzas and Shreve (1988)), for $q_1 = 1, 2, \dots$ we have

$$\mathbb{E}^0 \left[\sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s, X(s))dW^0(s) \right|^{2q_1} \right] \leq C_2(q_1) \mathbb{E}^0 \left[\left(\int_0^T \sigma^2(t, X(t))dt \right)^{q_1} \right], \quad (3.9)$$

for some constant $0 < C_2(q_1) < \infty$. But there exists a constant $C_3(\sigma, q_1, T) \in (0, \infty)$, such that

$$\begin{aligned} \mathbb{E}^0 \left[\left(\int_0^T \sigma^2(t, X(t))dt \right)^{q_1} \right] &\leq T^{q_1-1} \mathbb{E}^0 \left[\int_0^T \sigma^{2q_1}(t, X(t))dt \right] \\ &= T^{q_1-1} \int_0^T \mathbb{E}^0 [\sigma^{2q_1}(t, X(t))] dt \leq T^{q_1-1} C_3(\sigma, q_1, T) \left(1 + \int_0^T \mathbb{E}^0 [|X(t)|^{2q_1}] dt \right) \\ &\leq T^{q_1-1} C_3(\sigma, q_1, T) \left(1 + \int_0^T \mathbb{E}^0 \left[\sup_{0 \leq s \leq t} |X(s)|^{2q_1} \right] dt \right), \end{aligned} \quad (3.10)$$

where the second inequality comes from the inequality (2.8), the linear growth property of $\sigma(t, \cdot)$. Inequalities (3.8), (3.9) and (3.10) imply

$$\begin{aligned} \mathbb{E}^0 \left[\sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s, X(s))dW^0(s) \right|^{2q_1} \right] \\ \leq C_4(x_0, \sigma, \gamma, N, q_1, T) \left(1 + \int_0^T \mathbb{E}^0 \left[\sup_{0 \leq s \leq t} \left| \int_0^s \sigma(v, X(v))dW^0(v) \right|^{2q_1} \right] dt \right), \end{aligned} \quad (3.11)$$

for some constant $C_4(x_0, \sigma, \gamma, N, q_1, T) \in (0, \infty)$. Then, by the Gronwall inequality (e.g. page 287 of [18] Karatzas and Shreve (1988)), we know that

$$\mathbb{E}^0 \left[\sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s, X(s))dW^0(s) \right|^{2q_1} \right] < \infty, \quad (3.12)$$

hence by inequality (3.8) we have

$$\mathbb{E}^0 \left[\sup_{0 \leq t \leq T} |X(t)|^{2q_1} \right] < \infty. \quad (3.13)$$

The equation (2.11) and Assumption 2.1 (ii) imply that, for $0 \leq t \leq T$,

$$\tilde{L}(t; u) \leq L^{-1}(t; u) = \exp \left\{ \int_0^t \frac{b^2(s, X(s); u)}{\sigma^2(s, X(s))} ds \right\} \tilde{L}(t; u) \leq \exp\{C^2 T\} \tilde{L}(t; u), \quad (3.14)$$

where we have defined

$$\tilde{L}(t; u) := \exp \left\{ - \int_0^t \frac{b(s, X(s); u)}{\sigma(s, X(s))} dW^0(s) - \frac{1}{2} \int_0^t \frac{b^2(s, X(s); u)}{\sigma^2(s, X(s))} ds \right\} \quad (3.15)$$

and noted

$$\tilde{L}(t; u) = - \int_0^t \tilde{L}(s; u) \frac{b(s, X(s); u)}{\sigma(s, X(s))} dW^0(s). \quad (3.16)$$

Using Assumption 2.1 (ii) and the same arguments as those leading to (3.13), as well as the equations (3.3), (3.14) and (3.16), we can show

$$\mathbb{E}^0 \left[\sup_{0 \leq t \leq T} L^{q_2}(t; u) \right] < \infty \quad \text{and} \quad \mathbb{E}^0 \left[\sup_{0 \leq t \leq T} L^{-q_2}(t; u) \right] < \infty. \quad (3.17)$$

It follows from equations (3.5) and (3.17) that

$$\mathbb{E}^0 [R^{q_3}(T; u)] < \infty. \quad (3.18)$$

Taking an arbitrary \mathbf{F} -stopping time τ with values in $[0, T]$, by the Hölder inequality and by the estimates (3.13), (3.17) and (3.18), we get

$$\begin{aligned} & \mathbb{E}^0 \left[\sup_{0 \leq t \leq T} |X(t)|^{q_1} L^{q_2}(\tau; u) R^{q_3}(\tau; u) \right] \\ & \leq \left(\mathbb{E}^0 \left[\sup_{0 \leq t \leq T} |X(t)|^{2q_1} \right] \right)^{1/2} (\mathbb{E}^0 [L^{4q_2}(\tau; u)])^{1/4} (\mathbb{E}^0 [R^{4q_3}(\tau; u)])^{1/4} \\ & \leq \left(\mathbb{E}^0 \left[\sup_{0 \leq t \leq T} |X(t)|^{2q_1} \right] \right)^{1/2} \left(\mathbb{E}^0 \left[\sup_{0 \leq t \leq T} L^{4q_2}(t; u) \right] \right)^{1/4} (\mathbb{E}^0 [R^{4q_3}(T; u)])^{1/4} < \infty. \end{aligned} \quad (3.19)$$

To derive the uniform integrability of the family (3.7) from (3.19), we use the Cauchy-Schwartz and Chebyshev inequalities to get the estimate

$$\begin{aligned} & \sup_{\tau \in \mathbf{S}} \mathbb{E}^0 \left[\sup_{0 \leq t \leq T} |X(t)|^{q_1} L^{q_2}(\tau; u) R^{q_3}(\tau; u) \mathbf{1}_{\left\{ \sup_{0 \leq t \leq T} |X(t)|^{q_1} L^{q_2}(\tau; u) R^{q_3}(\tau; u) > A \right\}} \right] \\ & \leq \sup_{\tau \in \mathbf{S}} \left(\mathbb{E}^0 \left[\sup_{0 \leq t \leq T} |X(t)|^{2q_1} L^{2q_2}(\tau; u) R^{2q_3}(\tau; u) \right] \right)^{1/2} \\ & \quad \cdot \left(\mathbb{P}^0 \left(\sup_{0 \leq t \leq T} |X(t)|^{q_1} L^{q_2}(\tau; u) R^{q_3}(\tau; u) > A \right) \right)^{1/2} \\ & \leq \frac{1}{A} \sup_{\tau \in \mathbf{S}} \mathbb{E}^0 \left[\sup_{0 \leq t \leq T} |X(t)|^{2q_1} L^{2q_2}(\tau; u) R^{2q_3}(\tau; u) \right], \end{aligned} \quad (3.20)$$

which tends to zero as $A \rightarrow \infty$, on the strength of (3.19). ■

Lemma 3.2 For $0 \leq t \leq T$, $x \in \mathbb{R}$, $l = (l_0, l_1, \dots, l_m) \in \mathbb{R}^{m+1}$, and $r = (r_1, \dots, r_m) \in \mathbb{R}^m$, consider the function α defined as

$$\begin{aligned} \alpha(t, x, l, r) := & \left(\sum_{j=1}^m p_j l_j r_j + e^{-\lambda t} l_0 \right) \left(h(x) + \frac{1}{2} \xi''(x) \sigma^2(t, x) \right) \\ & + \left(\sum_{j=1}^m p_j l_j r_j b(t, x; \mu_j) + e^{-\lambda t} l_0 b(t, x; \mu_0) \right) \xi'(x), \end{aligned} \quad (3.21)$$

and the function β defined as

$$\beta(t, x, l, r, z) := \left(\sum_{j=1}^m p_j l_j r_j + e^{-\lambda t} l_0 \right) (\xi(x + \gamma(x, z)) - \xi(x) + \xi'(x) \gamma(x, z) + c(x, z)). \quad (3.22)$$

Then, for $0 \leq t \leq T$, we have

$$\begin{aligned} & \mathbb{E}^0 [Z(t) | \mathbf{F}(t)] \left(\int_0^t h(X(s)) ds + \xi(X(t)) + \sum_{\tau_i \leq t} c(X(\tau_i-), \zeta_i) \right) \\ &= M^0(t) + \int_0^t \alpha(s, X(s), L(s), R(s)) ds + \sum_{\tau_i \leq t} \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i), \end{aligned} \quad (3.23)$$

where $M^0(\cdot)$ is some square integrable $(\mathbb{P}^0, \mathbf{F})$ -martingale with $M^0(0) = \xi(x_0)$,

$$R(t) := (R(t; \mu_1) \cdots, R(t; \mu_m)), \quad (3.24)$$

and

$$L(t) := (L(t; \mu_0), L(t; \mu_1) \cdots, L(t; \mu_m)). \quad (3.25)$$

Proof: Applying Itô's formula for semimartingales with jumps, we get

$$\begin{aligned} & \mathbb{E}^0 [Z(t) | \mathbf{F}(t)] \left(\int_0^t h(X(s)) ds + \xi(X(t)) + \sum_{\tau_i \leq t} c(X(\tau_i-), \zeta_i) \right) \\ &= \xi(x_0) + \int_0^t \left(\int_0^{s-} h(X(u)) du + \xi(X(s-)) + \sum_{\tau_i \leq s-} c(X(\tau_i-), \zeta_i) \right) d\mathbb{E}^0 [Z(s) | \mathbf{F}(s)] \\ &+ \int_0^t \mathbb{E}^0 [Z(s-) | \mathbf{F}(s-)] \xi'(X(s-)) \sigma(t, X(t)) dW^0(t) \\ &+ \int_{0+}^t \alpha(s-, X(s-), L(s-), R(s-)) ds + \sum_{\tau_i \leq t} \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i). \end{aligned} \quad (3.26)$$

By change of variables and the continuity of Riemann integrals,

$$\int_{0+}^t \alpha(s-, X(s-), L(s-), R(s-)) ds = \int_0^t \alpha(s, X(s), L(s), R(s)) ds. \quad (3.27)$$

Define

$$\begin{aligned} M^0(t) &:= \xi(x_0) + \int_0^t \left(\int_0^{s-} h(X(u)) du + \xi(X(s-)) + \sum_{\tau_i \leq s-} c(X(\tau_i-), \zeta_i) \right) d\mathbb{E}^0 [Z(s) | \mathbf{F}(s)] \\ &+ \int_0^t \mathbb{E}^0 [Z(s-) | \mathbf{F}(s-)] \xi'(X(s-)) \sigma(t, X(t)) dW^0(t). \end{aligned} \quad (3.28)$$

Equations (3.26)-(3.28) imply that (3.23) holds. Substituting (3.4) into (3.28), we get

$$M^0(t) = \xi(x_0) + \int_0^t dW^0(s) \left[\mathbb{E}^0 [Z(s-) | \mathbf{F}(s-)] \xi'(X(s-)) \right. \\ \left. + \left(\sum_{j=1}^m p_j \left(\int_0^t \frac{L(s; \mu_0)}{L(s; \mu_j)} \lambda e^{-\lambda s} ds \right) L(t; \mu_j) \frac{b(t, X(t); \mu_j)}{\sigma(t, X(t))} \right) \left(\begin{array}{c} \int_0^{s-} h(X(u)) du \\ + \xi(X(s-)) \end{array} \right) \right. \\ \left. + e^{-\lambda t} L(t; \mu_0) \frac{b(t, X(t); \mu_0)}{\sigma(t, X(t))} \right) \left(+ \sum_{\tau_i \leq s-} c(X(\tau_i-), \zeta_i) \right) \Big].$$

By Lemma 3.1, $M^0(\cdot)$ is an integral of \mathbb{P}^0 -square integrable processes with respect to the $(\mathbb{P}^0, \mathbf{F})$ -Brownian motion $W^0(\cdot)$, hence $M^0(\cdot)$ is also a local $(\mathbb{P}^0, \mathbf{F})$ -martingale.

We need to show that $M^0(\cdot)$ is a $(\mathbb{P}^0, \mathbf{F})$ -martingale, not just a local martingale. It suffices to show that the family $\{M^0(\tau)\}_{\tau \in \mathbf{S}}$ is uniformly integrable under the probability measure \mathbb{P}^0 . By equations (3.2) and (3.23), $M^0(\cdot)$ can be expressed alternatively as

$$M^0(t) = \left(\sum_{j=1}^m \left(p_j L(t; \mu_j) \int_0^t \frac{L(s; \mu_0)}{L(s; \mu_j)} \lambda e^{-\lambda s} ds \right) + e^{-\lambda t} L(t; \mu_0) \right) \left(\begin{array}{c} \int_0^t h(X(s)) ds \\ + \xi(X(t)) \end{array} \right) \\ - \int_0^t \alpha(s, X(s), L(s), R(s)) ds - \sum_{\tau_i \leq t} \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i). \quad (3.29)$$

From the expressions (3.29), (3.21), (3.22), (2.8) and Assumption 2.2 (ii)(iii), we know that there exist a constant $C > 0$ and a positive integer q , such that

$$|M^0(t)| \\ \leq C \left(\begin{array}{c} \sum_{j=1}^m L(t; \mu_j) R(t; \mu_j) + L(t; \mu_0) \\ + \int_0^t \sum_{j=1}^m L(s; \mu_j) R(s; \mu_j) + L(s; \mu_0) ds \end{array} \right) \sup_{0 \leq s \leq T} |X(s)|^q, \quad (3.30)$$

for all $(t, \omega) \in [0, T] \times \Omega$. Then, from Lemma 3.1, we know that, under the probability measure \mathbb{P}^0 , the local martingale $M^0(\cdot)$ is both square-integrable and of class **D** on $[0, T]$. The latter implies that $M^0(\cdot)$ is a $(\mathbb{P}^0, \mathbf{F})$ -martingale. \blacksquare

Because the process $M^0(\cdot)$ in (3.23) and (3.28) is a $(\mathbb{P}^0, \mathbf{F})$ -martingale, it should vanish from inside the \mathbb{P}^0 -expectation of (3.1), leaving only the initial value and the finite variation part of the semimartingale. This property enables Lemma 3.3 to rewrite the \mathbb{P}^0 -expectations in (3.1) in a more convenient manner.

Lemma 3.3 *For any impulse control $(\tau, \zeta) \in \mathbf{I}$,*

$$\mathbb{E}^0 \left[\mathbb{E}^0 [Z(T) | \mathbf{F}(T)] \left(\int_0^T h(X(t)) dt + \xi(X(T)) + \sum_{i=1}^N c(X(\tau_i-), \zeta_i) \right) \right] \quad (3.31) \\ = \xi(x_0) + \mathbb{E}^0 \left[\int_0^T \alpha(s, X(s), L(s), R(s)) ds + \sum_{i=1}^N \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i) \right].$$

Proof: This is because the process

$$\begin{aligned} M^0(t) = \mathbb{E}^0 [Z(t) | \mathbf{F}(t)] & \left(\int_0^t h(X(s)) ds + \xi(X(t)) \right) - \int_0^t \alpha(s, X(s), L(s), R(s)) ds \\ & - \sum_{\tau_i \leq t} \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i), \quad 0 \leq t \leq T \end{aligned} \quad (3.32)$$

is a $(\mathbb{P}^0, \mathbf{F})$ -martingale by Lemma 3.2, hence $\mathbb{E}^0 [M^0(T)] = M^0(0) = \xi(x_0)$. Equivalently, equation (3.31) holds. \blacksquare

Up to this point, the \mathbb{P}^0 -expected reward from (3.1) has been rewritten into the \mathbb{P}^0 -expectation of the sum of a reward α cumulated over the time interval $[0, T]$, and of a reward β received only at the times of intervention, as in equation (3.31). Both α and β are functions of the processes $X(\cdot)$, $L(\cdot)$ and $R(\cdot)$, which are adapted to the observation filtration \mathbf{F} . Lemma 3.4 and Proposition 3.1 will show that the triple of processes $(X(\cdot), L(\cdot), R(\cdot))$ forms a well-behaved Markov process, because it is the unique strong solution to a stochastic differential equation with locally Lipschitz coefficients and this solution does not explode.

Lemma 3.4 *The triple $(X(\cdot), L(\cdot), R(\cdot))$ is a $(2m+2)$ -dimensional Markov process on every time interval $[\tau_i, \tau_{i+1})$, for $i = 0, 1, \dots, N-1$.*

Proof: Denoting $\mathbf{1} = (1, 1, \dots, 1)$ as the $(m+1)$ -dimensional row vector of 1's, and $\mathbf{0} = (0, \dots, 0)$ as the m -dimensional row vector of 0's. Over the time interval (τ_i, τ_{i+1}) , the triple $(X(\cdot), L(\cdot), R(\cdot))$ constitutes a strong solution to the $(2m+2)$ -dimensional SDE

$$\begin{cases} dX(t) = \sigma(t, X(t)) dW^0(t); \\ dL(t; \mu_j) = L(t; \mu_j) \frac{b(t, X(t); \mu_j)}{\sigma(t, X(t))} dW^0(t), \quad j = 0, 1, \dots, m; \\ dR(t; \mu_j) = \frac{L(t; \mu_0)}{L(t; \mu_j)} \lambda e^{-\lambda t} dt, \quad j = 1, \dots, m \end{cases} \quad (3.33)$$

driven by the standard \mathbb{P}^0 -Brownian motion $W^0(\cdot)$, with the initial value

$$(X(0), L(0), R(0)) = (x_0, \mathbf{1}, \mathbf{0}) \quad (3.34)$$

at time 0 and the initial value $(X(\tau_i), L(\tau_i), R(\tau_i))$ at the time τ_i . From Assumption 2.1 (i)(ii) and inequality (2.8), the coefficients of the SDE (3.33) are bounded on compact subsets of \mathbb{R}^{2m+2} and are locally Lipschitz. The SDE (3.33) has a pathwise unique, strong solution. The well-posedness of the SDE (3.33) (equivalently, the well-posedness of the associated martingale problem) implies the \mathbb{P}^0 -strong Markov property of $(X(\cdot), L(\cdot), R(\cdot))$, with respect to the Borel σ -algebra \mathbb{F} ([34] Stroock and Varadhan (1997)). But the filtration \mathbf{F} generated by $X(\cdot)$ is contained in \mathbb{F} , and the process $(X(\cdot), L(\cdot), R(\cdot))$ is \mathbf{F} -adapted. Then $(X(\cdot), L(\cdot), R(\cdot))$ has the strong Markov property under the probability measure \mathbb{P}^0 with respect to \mathbf{F} . \blacksquare

Proposition 3.1 *The solution $(X(\cdot), L(\cdot), R(\cdot))$ to the SDE (3.33) does not explode within the time horizon $[0, T]$.*

Proof: By the definition of the explosion time of an SDE with locally Lipschitz coefficients (e.g. page 330 of [18] Karatzas and Shreve (1988)), this follows from Lemma 3.1. ■

Eventually, we are able to reformulate in Theorem 3.1 the partially observable impulse control problem (2.17). Under the reference probability measure \mathbb{P}^0 , it becomes a fully observable impulse control problem of the $(2m + 2)$ -dimensional \mathbf{F} -adapted state process $(X(\cdot), L(\cdot), R(\cdot))$ from the SDE (3.33). To prove this theorem, we use equations (2.17), (3.1) and Lemma 3.3.

Theorem 3.1 *The impulse control problem (2.17) under the physical measure \mathbb{P} is equivalent to an impulse control problem under the reference probability measure \mathbb{P}^0 , by choosing an optimal $(\tau^*, \zeta^*) = \{(\tau_i^*, \zeta_i^*)\}_{i=1}^N$ to achieve the maximal expected reward*

$$V^0 := \sup_{(\tau, \zeta) \in \mathbf{I}} \mathbb{E}^0 \left[\int_0^T \alpha(s, X(s), L(s), R(s)) ds + \sum_{i=1}^N \beta(\tau_i, X(\tau_i-), L(\tau_i-), R(\tau_i-), \zeta_i) \right]. \quad (3.35)$$

Furthermore, the two maximal expected rewards are related by $V = \xi(x_0) + V^0$.

Because the best expected values V and V_0 are different only up to a constant $\xi(x_0)$, the two suprema in (2.17) and (3.35) are achieved by the same set of optimal control (τ^*, ζ^*) , if any. The impulse control problem (3.35) is the one we shall solve.

3.2 Solution under the reference probability measure

This subsection will solve the impulse control problem (3.35), by representing the optimal control (τ^*, ζ^*) in Proposition 3.2 in terms of the value function and the state process. The cornerstone of the representation is the dynamic programming principle of Lemma 3.5. To satisfy the technical condition of the Snell envelope argument for Proposition 3.2, the continuity of the value functions is provided in Lemma 3.6.

To save notations, some abbreviations are introduced first. We denote by \mathbf{O} the range of the solution $(X(\cdot), L(\cdot), R(\cdot))$, and its boundaries as

$$Q := [0, T] \times \mathbf{O} \quad \text{and} \quad \partial^* Q := \{T\} \times \mathbf{O}. \quad (3.36)$$

The state space \mathbf{O} differs for different parameters $b(\cdot, \cdot; u)$ and $\sigma(\cdot, \cdot)$. Without loss of generality, the variational inequalities associated with the impulse control problem shall be studied over the largest possible domain, which is

$$\mathbf{O} = \mathbb{R} \times (0, \infty)^{m+1} \times [0, \infty)^m. \quad (3.37)$$

For every $n = 1, 2, \dots$, denote the bounded domain

$$\mathbf{O}_n := (-n, n) \times \left(\frac{1}{n}, n\right)^{m+1} \times [0, n]^m \subset \mathbf{O} \subset \mathbb{R}^{2m+2}. \quad (3.38)$$

The closure of \mathbf{O}_n , denoted as $\bar{\mathbf{O}}_n$, is strictly contained in \mathbf{O} . As $n \rightarrow \infty$, the sets \mathbf{O}_n increase to \mathbf{O} , hence the sets $Q_n := [0, T] \times \mathbf{O}_n$ increase to Q . We introduce the abbreviations

$$y = (x, l_0, l_1, \dots, l_m, r_1, \dots, r_m), \quad (3.39)$$

$$b_Y(t, y) = \left(0, 0, 0, \dots, 0, \frac{l_0}{l_1} \lambda e^{-\lambda t}, \dots, \frac{l_0}{l_m} \lambda e^{-\lambda t}\right), \quad (3.40)$$

$$\sigma_Y(t, y) = \left(\sigma(t, x), l_0 \frac{b(t, x; \mu_0)}{\sigma(t, x)}, l_1 \frac{b(t, x; \mu_1)}{\sigma(t, x)}, \dots, l_m \frac{b(t, x; \mu_m)}{\sigma(t, x)}, 0, \dots, 0\right), \quad (3.41)$$

and

$$\Gamma(y, z) = (x + \gamma(x, z), l, r), \quad (3.42)$$

for all $(t, y) = (t, x, l, r)$ in Q . With this notation, the SDE (3.33), which has a pathwise unique, strong solution

$$Y(\cdot) = (X(\cdot), L(\cdot; \mu_0), L(\cdot; \mu_1), \dots, L(\cdot; \mu_m), R(\cdot; \mu_1), \dots, R(\cdot; \mu_m)), \quad (3.43)$$

can be written in the vector form

$$\begin{cases} dY(t) = b_Y(t, Y(t))dt + \sigma_Y(t, Y(t))dW^0(t), & \tau_i < t < \tau_{i+1}; \\ Y(\tau_i) = \Gamma(Y(\tau_i-), \zeta_i), & \text{for } i = 1, 2, \dots, N. \end{cases} \quad (3.44)$$

Here the initial value is

$$Y(0) = (x_0, \mathbf{1}, \mathbf{0}), \quad (3.45)$$

and $W^0(\cdot)$ is a standard \mathbb{P}^0 -Brownian motion.

In the abbreviated notation, the maximal expected reward in equation (3.46) can be written as

$$V^0 = \sup_{(\tau, \zeta) \in \mathbf{I}} \mathbb{E}^0 \left[\int_0^T \alpha(s, Y(s)) ds + \sum_{i=1}^N \beta(\tau_i, Y(\tau_i-), \zeta_i) \right]. \quad (3.46)$$

The rest of this section will use the above abbreviated notations.

Lemma 3.5 Dynamic Programming Principle. *For any $k \in \{1, 2, \dots, N\}$, and any $0 \leq t \leq T$, let $\mathbf{I}_{t,k}$ be the set of admissible interventions $\{(\tau_i, \zeta_i)\}_{i=N-k+1}^N$ such that $\tau_{N-k+1} \geq t$. Suppose the current value of the state process $Y(t) = y \in \mathbf{O}$. There exist deterministic measurable functions $v_0, v_1, \dots, v_N : Q \rightarrow \mathbb{R}$, such that*

$$v_k(t, y) = \operatorname{esssup}_{\{(\tau_i, \zeta_i)\}_{i=N-k+1}^N \in \mathbf{I}_{t,k}} \mathbb{E}^0 \left[\int_t^T \alpha(s, Y(s)) ds + \sum_{i=N-k+1}^N \beta(\tau_i, Y(\tau_i-), \zeta_i) \middle| \mathbf{F}(t) \right], \quad (3.47)$$

for $k = 1, \dots, N$, and

$$v_0(t, y) = \mathbb{E}^0 \left[\int_t^T \alpha(s, Y(s)) ds \middle| \mathbf{F}(t) \right]. \quad (3.48)$$

The value functions v_1, \dots, v_N satisfy the dynamic programming principle

$$\begin{aligned} v_k(t, y) = \operatorname{ess\,sup}_{(\tau_{N-k+1}, \zeta_{N-k+1}) \in \mathbf{I}_{t,1}} \mathbb{E}^0 \left[\int_t^{\tau_{N-k+1}} \alpha(s, Y(s)) ds \right. \\ \left. + \beta(\tau_{N-k+1}, Y(\tau_{N-k+1}-), \zeta_{N-k+1}) + v_{k-1}(\tau_{N-k+1}, \Gamma(Y(\tau_{N-k+1}), \zeta_{N-k+1})) \middle| \mathbf{F}(t) \right]. \end{aligned} \quad (3.49)$$

Proof: The existence of the functions v_0, v_1, \dots, v_N comes from the Markovian structure of the state process $Y(\cdot)$, by Lemma 3.4.

To prove the equation (3.49), fix an arbitrary $k \in \{1, 2, \dots, N\}$, an arbitrary $t \in [0, T]$ and arbitrary admissible interventions $\{(\tau_i, \zeta_i)\}_{i=N-k+1}^N \in \mathbf{I}_{t,k}$, we denote

$$\begin{aligned} A_k(t) &:= \int_t^{\tau_{N-k+1}} \alpha(s, Y(s)) ds + \beta(\tau_{N-k+1}, Y(\tau_{N-k+1}-), \zeta_{N-k+1}); \\ B_k &:= \int_{\tau_{N-k+1}}^T \alpha(s, Y(s)) ds + \sum_{i=N-k+2}^N \beta(\tau_i, Y(\tau_i-), \zeta_i). \end{aligned} \quad (3.50)$$

Then

$$\begin{aligned} &\mathbb{E}^0 \left[\int_t^T \alpha(s, Y(s)) ds + \sum_{i=N-k+1}^N \beta(\tau_i, Y(\tau_i-), \zeta_i) \middle| \mathbf{F}(t) \right] \\ &= \mathbb{E}^0 [A_k(t) + \mathbb{E}^0 [B_k | \mathbf{F}(\tau_{N-k+1})] | \mathbf{F}(t)]. \end{aligned} \quad (3.51)$$

On one hand, taking supremum over $(\tau_{N-k+1}, \zeta_{N-k+1}) \in \mathbf{I}_{t,1}$ on both sides of the inequality

$$\begin{aligned} &\mathbb{E}^0 [A_k(t) + \mathbb{E}^0 [B_k | \mathbf{F}(\tau_{N-k+1})] | \mathbf{F}(t)] \\ &\leq \mathbb{E}^0 [A_k(t) + v_{k-1}(\tau_{N-k+1}, \Gamma(Y(\tau_{N-k+1}), \zeta_{N-k+1})) | \mathbf{F}(t)] \end{aligned} \quad (3.52)$$

shows $v_k(t, y)$ less than or equal to the right hand side of (3.49). On the other hand, the inequality

$$v_k(t, y) \geq \mathbb{E}^0 [A_k(t) + \mathbb{E}^0 [B_k | \mathbf{F}(\tau_{N-k+1})] | \mathbf{F}(t)] \quad (3.53)$$

implies

$$v_k(t, y) \geq \mathbb{E}^0 [A_k(t) + v_{k-1}(\tau_{N-k+1}, \Gamma(Y(\tau_{N-k+1}), \zeta_{N-k+1})) | \mathbf{F}(t)] \quad (3.54)$$

and thus $v_k(t, y)$ greater than or equal to the right hand side of (3.49).

See [13] Fleming & Soner (1993), [21] Krylov (1980) or [30] Pham (2009) for a more detailed account for the dynamic programming principle. ■

Lemma 3.6 *The value functions v_0, v_1, \dots, v_N defined in (3.47) and (3.48) are continuous in $(t, y) \in Q$. Over the compact set $\bar{\mathbf{O}}_n$, they admit moduli of continuity $\omega_n : [0, \infty) \rightarrow [0, \infty)$, uniformly for all $0 \leq t \leq T$, meaning that*

$$|v_k(t, y^1) - v_k(t, y^2)| \leq \omega_n(\|y^1 - y^2\|), \quad \text{for all } (t, y^1), (t, y^2) \in [0, T] \times \bar{\mathbf{O}}_n. \quad (3.55)$$

Proof: By the continuity of solutions to SDEs (Theorem 5.2 in Chapter II on page 229 of [22] Kunita (1982)), and by the continuity of the function γ given in Assumption 2.2 (iii), the unique strong solution to the controlled SDE (3.33) is continuous in its initial value $(t, Y(t)) = (t, y) \in Q$. We shall also use the continuity of the functions α and β in equations (3.21) and (3.22), Assumption 2.2 (ii)(iii) and the uniform integrability Lemma 3.1. Inductively applying the proof of Proposition 2.2 in [16] Jalliet, Lamberton and Lapeyre (1990) to v_k , for $k = 0, 1, \dots, N$, we know that the value functions v_0, v_1, \dots, v_N are continuous in $(t, y) \in Q$.

Restricted on the compact set $\bar{Q}_n = [0, T] \times \bar{\mathbf{O}}_n$, the value functions v_0, v_1, \dots, v_N are uniformly continuous in $(t, y) \in \bar{Q}_n$, hence they admit moduli of continuity ω_n in the space variable $y \in \bar{\mathbf{O}}_n$, for all $t \in [0, T]$. ■

The collection of all continuous functions over the domain Q , which admit the modulus of continuity ω_n for y in the compact set $\bar{\mathbf{O}}_n$ uniformly for all $0 \leq t \leq T$, is denoted as $\mathbf{C}(Q; \omega_n)$. It is the very set of properties described in Lemma 3.6.

The optimal impulse controls are then obtained in terms of the value functions v_0, v_1, \dots, v_N , and of the triple $(X(\cdot), L(\cdot), R(\cdot)) = Y(\cdot)$. The triple $(X(\cdot), L(\cdot), R(\cdot)) = Y(\cdot)$ of processes in (3.33), which is adapted to the filtration \mathbf{F} generated by the observation $X(\cdot)$, can be viewed as a “sufficient statistic” for the optimization problem (2.17). This “sufficient statistic” that the decision maker needs to monitor remains the same for all cumulative reward functions $h(\cdot)$, all impulse control costs $c(\cdot)$ and all terminal reward functions $\xi(\cdot)$ in (2.17).

Proposition 3.2 *(Iterative procedure for optimization) For any measurable function $f : Q \rightarrow \mathbb{R}$, define a mapping \mathbf{M} by*

$$(\mathbf{M}f)(t, y) := \sup_{z \in \mathbb{R}} \{f(t, \Gamma(y, z)) + \beta(t, y, z)\}, \quad \text{for all } (t, y) \in Q. \quad (3.56)$$

For every $k = 1, 2, \dots, N$, iteratively define an \mathbf{F} -stopping time

$$\tau_k^* := \inf \left\{ \tau_{k-1}^* < t \leq T \mid v_{N-k+1}(t, Y(t)) \leq \mathbf{M}v_{N-k}(t, Y(t)) \right\}, \quad (3.57)$$

with the convention that $\tau_0^* = 0$. Suppose the supremum

$$\sup_{z \in \mathbb{R}} \{v_{N-k+1}(t, \Gamma(y, z)) + \beta(t, y, z)\} - v_{N-k}(t, y) \quad (3.58)$$

can be attained by a real number $z_k(t, y)$, and define an $\mathbf{F}(\tau_k^* -)$ -measurable random variable

$$\zeta_k^* := z_k(\tau_k^*, Y(\tau_k^* -)), \quad (3.59)$$

for every $k = 1, 2, \dots, N$. Then the suprema in (2.17) and (3.1) are attained by the set of impulse controls $\{\tau_k^*, \zeta_k^*\}_{k=1}^N$ in \mathbf{I} . Furthermore, the maximal expected rewards $V^0 = v_N(0, Y(0))$ and $V = \xi(x_0) + v_N(0, Y(0))$.

Remark 3.1 *When Bensoussan and Lions were originally formulating the impulse control problem in the 1970's, their number of interventions $N = \infty$. There is no fundamental difference whether N is finite or infinite, except that slightly different technical conditions on the coefficients and the admissible control set are required to derive properties like well-posedness, continuity and even differentiability of the value function. It has been pointed out by Bensoussan and Lions in Theorem 4 of [6] that the value function of N interventions converges to that of infinitely many interventions, as $N \rightarrow \infty$. To extend results in this paper to $N = \infty$ means modifying the technical assumptions.*

4 Geometric Brownian motion with drift uncertainty

In this part, we discuss how to approximate the optimal stopping time distribution $\mathbb{P}(\tau_i^* \in (t, t + dt])$ thanks to a Monte Carlo simulation, and use the results for constructing a trading strategy. Thus, we start by introducing in Section 4.1 the example on which we implement the theoretical setting of Sections 3.1 and 3.2. Then, we present in Section 4.2 a method based on Longstaff-Schwartz algorithm to simulate the optimal stopping times $\{\tau_i^*\}_{i=1,\dots,N}$ family. In Section 4.3, we give a simple static trading strategy that allows to test the simulation results.

4.1 Setting the problem

To illustrate the model (2.16), we discuss Geometric Brownian Motion as a commonly seen simple example. The parameter $\theta(\cdot)$ is the drift with the initial value μ_0 . The random variable U has the prior distribution

$$U = \begin{cases} \mu_1, & \text{with probability } p_1; \\ \mu_2, & \text{with probability } p_2 = 1 - p_1 \end{cases} \quad (4.1)$$

and ρ has an exponential λ prior distribution as in (2.4). The diffusion $X(\cdot)$ in (2.16) is the geometric Brownian motion

$$\begin{cases} dX(t) = X(t)\theta(t)dt + X(t)\sigma dW(t); \\ X(0) = x_0. \end{cases} \quad (4.2)$$

In this example, the volatility σ is a deterministic positive number. The parameter $\theta(\cdot)$ with the initial value μ_0 is the percentage drift of the Geometric Brownian motion.

Suppose $X(\cdot)$ is the price process of a certain stock, and there is zero interest rate, no transaction cost and no price impact. Observing the price evolution only, an optimal trading problem is finding two stopping times $0 \leq \tau_1^* \leq \tau_2^* \leq T$ in \mathbf{S} , to achieve the supremum in

$$\sup_{\tau_1 \text{ and } \tau_2 \in \mathbf{S}, \tau_1 \leq \tau_2} \mathbb{E}[X(\tau_2) - X(\tau_1)]. \quad (4.3)$$

In terms of the money received, the value (4.3) is the best possible average profit from first buying, then selling, one share of this stock. Comparing the SDEs (4.2) and (2.16), and the goal (2.17) with (4.3), we are trying to solve the impulse control problem with $\gamma(\cdot, \cdot) = 0$, $h(\cdot) = 0$ and $\xi(\cdot) = 0$. We should set $c(x, z) = zx$ for $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$, $\zeta_1 = -1$, $\zeta_2 = 1$ and $N = 2$. To incorporate transaction costs and price impact, it only remains to modify the functions $c(\cdot, \cdot)$ and $\gamma(\cdot, \cdot)$.

For the geometric Brownian motion example, we may compute to get the likelihood ratio processes

$$L(t; u) = \exp \left\{ \frac{1}{2} \left(u - \frac{u^2}{\sigma^2} \right) t \right\} \left(\frac{X(t)}{x_0} \right)^{u/\sigma^2}, \quad 0 \leq t \leq T. \quad (4.4)$$

The process $R(\cdot; u)$ defined in (3.5) can then be written as

$$R(t; u) = \int_0^t \lambda \exp \left\{ \left(\frac{1}{2} \left(\mu_0 - u - \frac{\mu_0^2 - u^2}{\sigma^2} \right) - \lambda \right) s \right\} \left(\frac{X(s)}{x_0} \right)^{(\mu_0 - u)/\sigma^2} ds, \quad 0 \leq t \leq T, \quad (4.5)$$

for $u \in \Theta$. The dimensionality of the variational inequality can be reduced, by using this alternative expression of $L(t; u)$ in terms of $X(t)$. Substituting $l_j = \exp \left\{ \frac{1}{2} \left(\mu_j - \frac{\mu_j^2}{\sigma^2} \right) t \right\} \left(\frac{x}{x_0} \right)^{\mu_j/\sigma^2}$, $\gamma(\cdot, \cdot) = 0$, $h(\cdot) = 0$, $\xi(\cdot) = 0$ and $c(x, z) = zx$ in two the functions α and β defined in (3.21) and (3.22), we get

$$\alpha(t, x, l, r) = 0, \quad (4.6)$$

$$\begin{aligned} \beta(t, x, l, r, z) &= z \left(\sum_{j=1}^2 p_j \exp \left\{ \frac{1}{2} \left(\mu_j - \frac{\mu_j^2}{\sigma^2} \right) t \right\} \frac{x^{1+\mu_j/\sigma^2}}{x_0^{\mu_j/\sigma^2}} r_j \right. \\ &\quad \left. + \exp \left\{ \frac{1}{2} \left(\mu_0 - \frac{\mu_0^2}{\sigma^2} \right) t - \lambda t \right\} \frac{x^{1+\mu_0/\sigma^2}}{x_0^{\mu_0/\sigma^2}} \right) \\ &=: \bar{\beta}(t, x, r, z). \end{aligned} \quad (4.7)$$

Under the measure \mathbb{P}^0 , the supremum in (4.3) becomes

$$\begin{aligned} &\sup_{\tau_1 \text{ and } \tau_2 \in \mathbf{S}, \tau_1 \leq \tau_2} \mathbb{E} [X(\tau_2) - X(\tau_1)] \\ &= \sup_{\tau_1 \text{ and } \tau_2 \in \mathbf{S}, \tau_1 \leq \tau_2} \mathbb{E}^0 [\bar{\beta}(\tau_2, X(\tau_2), R(\tau_2), 1) + \bar{\beta}(\tau_1, X(\tau_1), R(\tau_1), -1)]. \end{aligned} \quad (4.8)$$

There exist deterministic measurable functions \bar{v}_1 and $\bar{v}_2 : [0, T] \times (0, \infty) \times [0, \infty)^2$, such that

$$\begin{aligned} \bar{v}_1(t, X(t), R(t)) &= \sup_{\tau_2 \in \mathbf{S}_t} \mathbb{E}^0 [\bar{\beta}(\tau_2, X(\tau_2), R(\tau_2), 1) | \mathbf{F}(t)]; \\ \bar{v}_2(t, X(t), R(t)) &= \sup_{\tau_1 \in \mathbf{S}_t} \mathbb{E}^0 [\bar{\beta}(\tau_1, X(\tau_1), R(\tau_1), -1) + \bar{v}_1(\tau_1, X(\tau_1), R(\tau_1)) | \mathbf{F}(t)]. \end{aligned} \quad (4.9)$$

The optimal value of the round-way transaction is

$$\sup_{(\tau_1, \tau_2) \in \mathbf{S}^2, \tau_1 \leq \tau_2} \mathbb{E} [X(\tau_2) - X(\tau_1)] = \bar{v}_2(0, X(0), \mathbf{0}). \quad (4.10)$$

4.2 Longstaff-Schwartz algorithm for multiple optimal stopping times

Proposed in [27] for pricing American options, the Longstaff-Schwartz procedure was rigourously formulated by Clément, Lamberton and Protter (2002) in [9] using stopping times instead of the value function for the dynamic programming algorithm. This method also involves a regression approximation of the conditional expectation and the discretization of the interval on which the stopping times take their values. The convergence due to each approximation step is studied in [9] and we aim at reusing this algorithm for our multiple optimal stopping problem.

Applied also by Tsitsiklis and Van Roy (2001) in [35], the regression approximation of the conditional expectation was extended to BSDEs (Backward Stochastic Differential Equations) by Gobet, Lemor and Warin (2005) in [15]. The regression approximation generally uses one regression vector for the whole set of trajectories and it is then considered as a global method. Thus, other authors use more local approximations of the conditional expectation, based on either Malliavin calculus as in [1, 8] or quantization method as in [2]. To keep the presentation of our algorithm simple, we apply a monomial regression method. However, one has to keep in mind that in some problems, especially when the dimension becomes high (more than two assets), a good approximation of the conditional expectation is a key ingredient.

To proceed, we first need to approach stopping times in \mathbf{S} with stopping times taking values in the finite set $0 = t_0 < t_1 < \dots < t_n = T$. Then, the computation of (4.9) can be reduced to the implementation of two dynamic programming algorithms that we express in terms of the optimal stopping times τ_1^k and τ_2^k , for each path, as follows

$$\begin{aligned} \tau_2^n &= T, \\ \text{for } k \in \{n-1, \dots, 1\}, \quad \tau_2^k &= t_k 1_{A_{1k}} + \tau_2^{k+1} 1_{A_{1k}^c}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \tau_1^n &= T, \\ \text{for } k \in \{n-1, \dots, 1\}, \quad \tau_1^k &= t_k 1_{A_{2k}} + \tau_2^k \wedge \tau_1^{k+1} 1_{A_{2k}^c} \end{aligned} \tag{4.12}$$

and, denoting $\mathbb{E}_{t_k}^0$ the conditional expectation knowing $\mathbf{F}(t_k)$, the sets A_{1k} and A_{2k} are given by

$$\begin{aligned} A_{1k} &= \left\{ \bar{\beta}(t_k, X(t_k), R(t_k), 1) > \mathbb{E}_{t_k}^0[\bar{v}_1(t_{k+1}, X(t_{k+1}), R(t_{k+1}))] \right\}; \\ A_{2k} &= \left\{ \begin{array}{l} \bar{\beta}(t_k, X(t_k), R(t_k), -1) \\ + \bar{v}_1(t_k, X(t_k), R(t_k)) \end{array} > \mathbb{E}_{t_k}^0[\bar{v}_2(t_{k+1}, X(t_{k+1}), R(t_{k+1}))] \right\}. \end{aligned}$$

While the simulation of X under \mathbb{P}^0 is straightforward, the simulation of R is performed thanks to a trapezoidal approximation of the time integral applied in [25] for Asian options. It remains then to compute $\mathbb{E}_{t_k}^0[\bar{v}_i(t_{k+1}, X(t_{k+1}), R(t_{k+1}))]$ for $i = 1, 2$. Employing the Markov property established in Lemma 3.4, a conditional expectation according to $\mathbf{F}(t_k)$ can be replaced by a conditional expectation according to

$Y(t_k) = (X(t_k), R(t_k))$. In our application, this latter quantity will be approximated by a regression on the monomial family $g(Y(t_k)) = (1, X(t_k), R(t_k, \mu_1), R(t_k, \mu_2))$. Formally, for the auxiliary functions $f_i(\cdot) = \bar{v}_i(t_k, \cdot, \cdot)$, $i = 1, 2$

$$\mathbb{E}^0(f_i(Y(t_{k+1}))|Y(t_k)) \approx A^i \cdot g(Y(t_k)). \quad (4.13)$$

The vector A^i minimizes the quadratic error

$$\|f_i(Y(t_{k+1})) - A^i \cdot g(Y(t_k))\|_{L^2} \quad (4.14)$$

and thus equal to

$$A^i = \Psi^{-1} \mathbb{E}^0(f_i(Y(t_{k+1}))g(Y(t_k))), \quad (4.15)$$

where the matrix $\Psi = \mathbb{E}^0(g(Y(t_k))g^t(Y(t_k)))$ and t is the transpose operator. Consequently, at each time step, the matrix inversion (4.15) can be implemented by the Singular Value Decomposition (SVD) explained in [31] and the expectations are approximated by an arithmetic average

$$\mathbb{E}^0(f_i(Y(t_{k+1}))g(Y(t_k))) \approx \frac{1}{M} \sum_{l=1}^M f_i(Y^l(t_{k+1}))g(Y^l(t_k)),$$

$$\mathbb{E}^0(g(Y(t_k))g^t(Y(t_k))) \approx \frac{1}{M} \sum_{l=1}^M g(Y^l(t_k))g^t(Y^l(t_k)).$$

and M is the number of simulated trajectories. Then, using τ_1^1 known from (4.12)

$$\bar{v}_2(0, X(0), 0) \approx \max(\mathbb{E}^0[\bar{v}_2(\tau_1^1, X(\tau_1^1), R(\tau_1^1))], 0) \quad (4.16)$$

is the approximation of (4.10). Also, for $i = 1, 2$ we make the approximation

$$\mathbb{P}^0(\tau_i^* \in (t_k, t_{k+1}]) \approx \mathbb{P}^0(\tau_i^1 = t_{k+1}). \quad (4.17)$$

Finally, we should point out that the proposed procedure can be generalized for more than two optimal stopping times. The convergence of the overall algorithm can be established in the same way as it is presented in [9] for one optimal stopping time. Besides, the reader should notice that we only considered the deterministic case $\zeta_1 = -1$, $\zeta_2 = 1$ and $N = 2$. In fact, one can propose a randomized version of the algorithm proposed above that includes an optimization over ζ , however our purpose here is only to give an illustration of a simple case. As a future work, we will study the convergence of a more general method for impulse control based on the multiple optimal stopping times algorithm presented above.

4.3 A trading strategy based on $\mathbb{P}(\tau_i^* \in (t, t + dt])$

To present this trading strategy, we need first to change the probability measure and go back to \mathbb{P} thanks to (2.13) and the simulation of $Z(t)$ using (2.12). For $i = 1, 2$, we obtain then the approximation

$$\mathbb{P}(\tau_i^* \in (t_k, t_{k+1}]) \approx \mathbb{E}^0 \left(Z(t_{k+1}) 1_{\tau_i^1 = t_{k+1}} \right). \quad (4.18)$$

Now, let us assume that we can buy and sell not only one stock but a bigger volume $q \geq 1$ of stocks. Consequently, one can use the approximation

$$q \sup_{\substack{(\tau_1, \tau_2) \in \mathbf{S}^2 \\ \tau_1 \leq \tau_2}} \mathbb{E} [X(\tau_2) - X(\tau_1)] \approx q \max \left(\mathbb{E}^0 [\bar{v}_2(\tau_1^1, X(\tau_1^1), R(\tau_1^1))] , 0 \right). \quad (4.19)$$

Using the value obtained in (4.19), we decide at a first stage if it is interesting to trade or not. Indeed, if this value is not big “enough” then it is not worthwhile taking the trading risks of losing money. In this one dimensional example, one should invest on X only if it is drifting more positively than negatively. Besides, if we are satisfied by the expected profits, we can establish the following static trading strategy on the spot prices $X(t_{k+1})$ for $k = 0, \dots, n - 1$: The money received is

$$M_{k+1}^{r*} = M_k^{r*} + q [\mathbb{P}(\tau_2^* \in (t_k, t_{k+1}]) - \mathbb{P}(\tau_1^* \in (t_k, t_{k+1}])] X(t_{k+1}), \quad M_0^{r*} = 0. \quad (4.20)$$

M^{r*} can take negative values which mean that we are buying stocks. We are going to compare M_n^{r*} to M_n^r where M_{k+1}^r is defined by

$$M_{k+1}^r = M_k^r + q [P_{k+1}^2 - P_{k+1}^1] X(t_{k+1}), \quad M_0^r = 0. \quad (4.21)$$

and the quantities P_{k+1}^1, P_{k+1}^2 are simulated thanks to the following increasing induction on $k = 0, \dots, n - 1$

$$\begin{aligned} P_{k+1}^1 &\sim U(0, 1 - S_{k+1}^1), & P_{k+1}^2 &\sim U(0, S_{k+1}^2), & P_n^2 &= S_n^2, \\ S_{k+1}^1 &= S_k^2 - P_k^2, & S_{k+1}^2 &= S_{k+1}^1 + P_{k+1}^1, & S_1^1 &= 0 \end{aligned} \quad (4.22)$$

where $U(0, x)$ is the uniform law on $[0, x]$.

Consequently, we are going to compare the money earned from our static trading strategy (4.18)(4.20) to some M_a^s arbitrary strategies specified by (4.21)(4.22). We process this comparison on a large number M^{new} (given later) of newly simulated trajectories of X under the probability \mathbb{P} . In the following, we denote respectively by \widetilde{M}_n^{r*} and \widetilde{M}_n^r the average value of M_n^{r*} and M_n^r on the M^{new} newly simulated trajectories of X . Also we denote by \overline{M}_n^{r*} and \overline{M}_n^r the maximum value of respectively M_n^{r*} and M_n^r on the M^{new} newly simulated trajectories of X .

Although we tested our algorithm for a large number of model parameters, we present here the results associated to only one choice of values. We refer the reader to the first author web page to download the C++ code of the algorithm in order to test it with other parameters values. Figures 1, 2 and 3 involve the following choice: Number of simulated trajectories for Longstaff-Schwartz algorithm $M = 2^{16}$, number

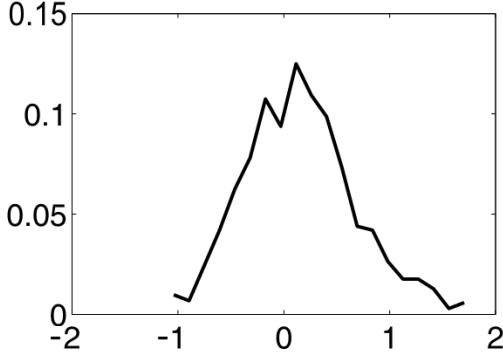


Figure 1: The histogram of \widetilde{M}_n^r associated to $M_a^s = 2^{10}$ arbitrary strategies. The static optimal strategy provides $\widetilde{M}_n^{r*} = 6.17$.

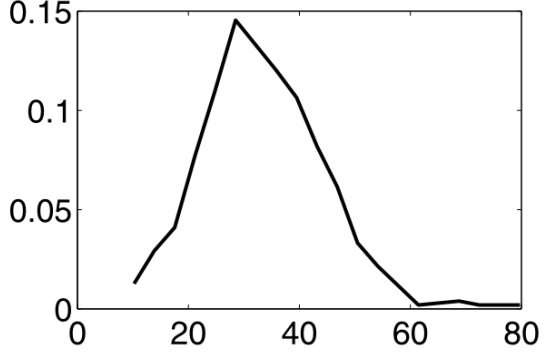


Figure 2: The histogram of \overline{M}_n^r associated to $M_a^s = 2^{10}$ arbitrary strategies. The static optimal strategy provides $\overline{M}_n^{r*} = 51.98$.

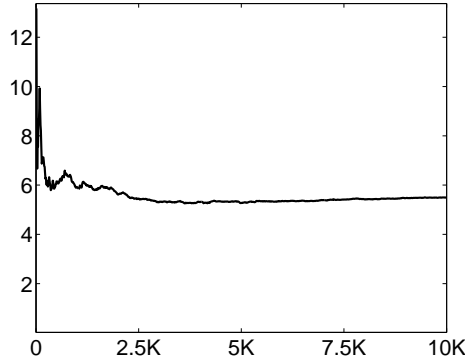


Figure 3: The evolution of \widetilde{M}_n^{r*} according to the number of simulated trajectories M^{new} (in kilo)

of time steps $n = 10$, $T = 1$, $\mu_0 = \mu_1 = 0.1$, $\mu_2 = -0.1$, $p_1 = 0.5$, $\sigma = 0.2$, $\lambda = 1$, $x_0 = 1$ and $q = 100$.

Using $M^{\text{new}} = 2^{10}$ new scenarios of the evolution of X , we compare in figures 1 and 2 the average profit as well as the maximum profit generated by the $M_a^s = 2^{10}$ arbitrary strategies to the ones generated by the static optimal strategy. In Figure 1, the optimal strategy outperforms all the arbitrary strategies which confirms the effectiveness of the method implemented in Section 4.2. Moreover, even the maximum profit provided by the optimal strategy is among the best according to Figure 2. In Figure 3, we show the stability of the optimal strategy to reach the average value of profits even for small numbers of scenarios $M^{\text{new}} \sim 200$.

To conclude this section, although one can establish more elaborate trading strategies using the approximated values of $\mathbb{P}(\tau_i^* \in (t, t + dt])$, the one that we provided in this section allowed us to show the efficiency of the Longstaff-Schwartz algorithm for multiple optimal stopping times.

5 Discussions

In Section 5.1, we discuss how the change of measure method can be extended to the multidimensional case. In Section 5.2, we briefly present the posterior probabilities method and explain what make it difficult to apply to the multidimensional partially-observed control problems.

5.1 The measure change method for multidimensional state processes

The measure change method proposed in Section 3 can be extended to the case when the diffusion in Section 2.1 is multidimensional, mostly by replacing the notations for scalars to those for matrices. This subsection will give the multidimensional version of the formulae whose modifications are not very straightforward.

Suppose the diffusion $X(\cdot)$ in equation (2.9) becomes a $(d \times 1)$ -dimensional process driven by a $(d \times 1)$ -dimensional Brownian motion $W^0(\cdot)$ with independent components. Correspondingly, the coefficients should be modified according to the dimensionality.

For any possible values $u \in \Theta$, the drift $b(\cdot, \cdot; u) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a mapping valued in \mathbb{R}^d and the volatility $\sigma(\cdot, \cdot) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is a $(d \times d)$ -matrix-valued mapping. Let $\|\cdot\|$ denote the Euclidean norm. Assumption 2.1 is replaced by Assumption 5.1.

Assumption 5.1 *There exists a constant $C > 0$, such that*
(i) for all $(t, x^1), (t, x^2) \in [0, T] \times \mathbb{R}^d$, and for all $u \in \Theta$, we have

$$\begin{aligned} & \|b(t, x^1; u) - b(t, x^2; u)\| + \|\sigma(t, x^1) - \sigma(t, x^2)\| \\ & + \left\| \frac{b(t, x^1; u)}{\sigma(t, x^1)} - \frac{b(t, x^2; u)}{\sigma(t, x^2)} \right\| \leq C \|x^1 - x^2\|; \end{aligned} \quad (5.1)$$

(ii) for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and all $u \in \Theta$, the matrix $\sigma(t, x)$ is invertible and

$$\left\| \frac{b(t, x; u)}{\sigma(t, x)} \right\| \leq C. \quad (5.2)$$

The reward functions ξ and $h : \mathbb{R}^d \rightarrow \mathbb{R}$, as well as the intervention impact γ and the reward from intervention $c : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$, satisfy Assumption 5.2 instead of Assumption 2.2.

Assumption 5.2 *(i) The function $\xi(\cdot)$ is twice continuously differentiable, with first and second order derivatives denoted as $\frac{\partial}{\partial x_i} \xi(\cdot)$ and $\frac{\partial^2}{\partial x_i \partial x_j} \xi(\cdot)$, for $i, j = 1, \dots, d$.*
(ii) The functions $h(\cdot)$, $\xi(\cdot)$, $\frac{\partial}{\partial x_i} \xi(\cdot)$ and $\frac{\partial^2}{\partial x_i \partial x_j} \xi(\cdot)$ are locally Lipschitz and have polynomial growth, for $i, j = 1, \dots, d$.
(iii) The function $\gamma(x, z)$ is bounded for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}$, and the function $c(x, z)$ has polynomial growth rate in $x \in \mathbb{R}^d$ uniformly for all $z \in \mathbb{R}$. Both functions $\gamma(x, z)$ and $c(x, z)$ are continuous in x , for any arbitrarily fixed $z \in \mathbb{R}$.

To change between the physical measure \mathbb{P} and the reference probability measure \mathbb{P}^0 , the likelihood ratio process (2.11) is replaced by

$$L(t; u) = \exp \left\{ \int_0^t (\sigma^{-1}(s, X(s))b(s, X(s); u))^t dW^0(s) - \frac{1}{2} \int_0^t \|\sigma^{-1}(s, X(s))b(s, X(s); u)\|^2 ds \right\}, \quad 0 \leq t \leq T. \quad (5.3)$$

where t is the transpose operator. Then, by the same derivation as in Section 3.1, we arrive at the impulse control problem (3.35) under the reference probability measure \mathbb{P}^0 . Let ∇ denote the gradient operator of a function. Instead of equations (3.21) and (3.22), the reward functions α and β in (3.35) are defined as

$$\begin{aligned} \alpha(t, x, l, r) := & \left(\sum_{j=1}^m p_j l_j r_j + e^{-\lambda t} l_0 \right) \left(h(x) + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^t)_{i,j}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} \xi(x) \right) \\ & + \left((\nabla \xi)(x) \sum_{j=1}^m p_j l_j r_j b(t, x; \mu_j) + e^{-\lambda t} l_0 b(t, x; \mu_0) \right), \end{aligned} \quad (5.4)$$

and

$$\beta(t, x, l, r, z) := \left(\sum_{j=1}^m p_j l_j r_j + e^{-\lambda t} l_0 \right) \left(\begin{aligned} & \xi(x + \gamma(x, z)) - \xi(x) \\ & + (\nabla \xi)(x) \gamma(x, z) + c(x, z) \end{aligned} \right). \quad (5.5)$$

The solution to the impulse control problem (3.35) with the multidimensional $X(\cdot)$ process will follow exactly the same steps as in Section 3.2.

5.2 Partial observation control via posterior probabilities

The traditional method to reduce a partially-observed control problem to one with full observation is to augment the state process $X(\cdot)$ by the posterior probability processes

$$\Pi_i(t) := \mathbb{P}(\theta(t) = \mu_i | \mathbf{F}(t)), \text{ for } i = 0, 1, \dots, m. \quad (5.6)$$

This method is presented in Section 2.4.6 of [29] Pham(2005) for a survey of the control problem and in Chapter 9 of [26] Lipster and Shiryaev (2001) for the derivation of the posterior expectation and probabilities. In this subsection, we shall first outline how to solve our problem in dimension one by the posterior probability method, modulus technical assumptions, and then briefly explore the relation between the two methods.

Define a function $\bar{b} : [0, T] \times \mathbb{R} \times [0, 1]^{m+1} \rightarrow \mathbb{R}$, $(t, x, \pi) \mapsto \bar{b}(t, x, \pi)$, by $\bar{b}(t, x, \pi) := \sum_{i=0}^m \pi_i b(t, x; \mu_i)$. Then the uncertain drift projected onto the observation filtration is

$$\mathbb{E}[b(t, X(t); \theta(t)) | \mathbf{F}(t)] = \bar{b}(t, X(t), \Pi(t)). \quad (5.7)$$

Let \bar{W} be the innovation Brownian motion. Given an arbitrary admissible impulse control $(\tau, \zeta) \in \mathbf{I}$, the augmented state process $(X(\cdot), \Pi(\cdot))$ is a $(m+2)$ -dimensional Markov process on every time interval $[\tau_k, \tau_{k+1})$, for $k = 0, 1, \dots, N-1$, because it is the unique strong solution to the controlled SDE

$$\begin{cases} X(t) = x_0 + \int_0^t \bar{b}(s, X(s), \Pi(s)) ds + \int_0^t \sigma(s, X(s)) d\bar{W}(s) + \sum_{\tau_i \leq t} \gamma(X(\tau_i-), \zeta_i); \\ \Pi_0(t) = 1 - \lambda \int_0^t \Pi_0(s) ds + \int_0^t \frac{b(s, X(s); \mu_0) - \bar{b}(s, X(s), \Pi(s))}{\sigma(s, X(s))} \Pi_0(s) d\bar{W}(s); \\ \Pi_i(t) = p_i \lambda \int_0^t \Pi_i(s) ds + \int_0^t \frac{b(s, X(s); \mu_i) - \bar{b}(s, X(s), \Pi(s))}{\sigma(s, X(s))} \Pi_i(s) d\bar{W}(s), \\ i = 1, \dots, m, 0 \leq t \leq T. \end{cases} \quad (5.8)$$

One can then use the state process $(X(\cdot), \Pi(\cdot))$ to solve the impulse control problem (2.17) under the physical measure, as a problem of full observation. The optimal impulse controls are represented through the routine dynamic programming arguments in terms of the value functions and the state process.

Let us proceed to demonstrate that the posterior probability method and the measure change method are theoretically equivalent. Comparing with those in Lemma 3.5, there exist value functions $u_0, u_1, \dots, u_N : [0, T] \times \mathbb{R} \times [0, 1]^{m+1} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} u_k(t, x, \pi) = \operatorname{esssup}_{\{(\tau_i, \zeta_i)\}_{i=N-k+1}^N \in \mathbf{I}_{t,k}} \mathbb{E} \left[\int_t^T h(s, X(s)) ds + \xi(X(T)) \right. \\ \left. + \sum_{i=N-k+1}^N c(X(\tau_i-), \zeta_i) \middle| \mathbf{F}(t) \right], \end{aligned} \quad (5.9)$$

for $k = 1, \dots, N$, and

$$u_0(t, x, \pi) = \mathbb{E} \left[\int_t^T h(s, X(s)) ds + \xi(X(T)) \middle| \mathbf{F}(t) \right]. \quad (5.10)$$

By the same reasoning that derives Theorem 3.1, the two sets of value functions respectively from the posterior probability method and the measure change method are related by the equations

$$u_k(t, x, \pi) = \xi(x) + v_k(t, x, l, r), \quad k = 0, 1, \dots, N, \quad (5.11)$$

for all $x \in \mathbb{R}$, $\pi \in [0, 1]^{m+1}$, $l \in (0, \infty)^{m+1}$ and $r \in [0, \infty)^m$. In the case where the conditions in the Implicit Mapping Theorem are satisfied, there exists an implicit mapping $\bar{\pi} : Q \rightarrow [0, 1]^{m+1}$, $(t, x, l, r) \mapsto \bar{\pi}(t, x, l, r)$, such that

$$u_k(t, x, \bar{\pi}(t, x, l, r)) = \xi(x) + v_k(t, x, l, r), \quad k = 0, 1, \dots, N, \quad (5.12)$$

for all $x \in \mathbb{R}$, $l \in (0, \infty)^{m+1}$ and $r \in [0, \infty)^m$. The expression (5.12) suggests that, when applicable, the value functions before and after the change of measure are different up to a change of variable.

Despite of the above equivalence, the measure change method has an advantage in several dimensions when it comes to the numerical implementation of Monte Carlo. Indeed, even when X is one-dimensional, one can easily remark that the Monte Carlo simulation of (3.33) is easier to perform and study than the simulation of (5.8). With the latter SDEs system, one has to propose an efficient discretization scheme and prove its convergence with an error control. However, this is not standard even when $d = 1$. Unlike (5.8), with (3.33), one needs only to use some usual methods of simulating diffusions as the ones presented in [14] for X , then simulate L and R as deterministic functionals of X . When both X and the Brownian motion are $(d \times 1)$ -dimensional processes, the contrast between the two methods become clearer. Add to this the complexity of studying how the discretization error of (5.8) effects the Longstaff-Schwartz multiple optimal stopping algorithm proposed in Section 4.2.

To conclude this section, we would like to point out that the method based on posterior probabilities is theoretically equivalent to the change of probability method. Moreover, to solve the problem when $d = 1$, one can use some discretization and weak convergence for both methods (We refer to [24] for more details). Nevertheless, when implementing an algorithm based on Monte carlo as Longstaff-Schwartz, the use of the change of probability is more appropriate and could be the method by default when $d > 1$.

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